Conformal Algebra

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Conformal Transformations

A conformal transformation is a smooth, invertible map $x \to x'$ which locally preserves the angle between any two lines:

$$g'_{\rho\sigma}(x') \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu}(x), \tag{1.1}$$

Let us next consider the infinitesimal coordinate transformations ($\epsilon(x) << 1$):

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + \mathcal{O}(\epsilon^2). \tag{1.2}$$

Conformal Transformations

For $d \geq 3$, there are ONLY 4 calsses of solutions for $\epsilon_\mu(x)$ of $x'_\mu = x_\mu + \epsilon_\mu(x) + \mathcal{O}(\epsilon^2)$.

$$\text{(Infinitesimal Translation)} \quad \epsilon^\mu(x) = a^\mu \quad \text{(constant)} \tag{1.3}$$

$$\text{(Infinitesimal Rotation)} \quad \epsilon^\mu(x) = M^\mu_\nu x^\nu \tag{1.4}$$

$$\text{(Infinitesimal Scaling)} \quad \epsilon^\mu(x) = \lambda x^\mu \tag{1.5}$$

$$\text{(Infinitesimal SCT)} \quad \epsilon^\mu(x) = 2(b.x)x^\mu - x^2 b^\mu \tag{1.6}$$

The Finite conformal transformations are:

$$\begin{aligned}
\text{(translation)} \quad & x'^\mu = x^\mu + a^\mu \\
\text{(rotation)} \quad & x'^\mu = M^\mu{}_\nu x^\nu \\
\text{(dilatation)} \quad & x'^\mu = \alpha x^\mu \\
\text{(SCT)} \quad & x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}
\end{aligned}$$

SCT

Inversions are given by

$$\boxed{x^\mu} \longrightarrow \boxed{x'^\mu = \frac{x^\mu}{x^2}} \tag{1.7}$$

The SCTs can be understood as an inversion of x^μ, followed by a translation b^μ, and followed again by an inversion.

$$\boxed{x^\mu} \longrightarrow \boxed{x'^\mu = \frac{x^\mu}{x^2}} \longrightarrow \boxed{x''^\mu = \frac{x^\mu}{x^2} - b^\mu} \longrightarrow \boxed{x'''^\mu = \frac{\frac{x^\mu}{x^2} - b^\mu}{(\frac{x^\mu}{x^2} - b^\mu)^2} = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}}$$

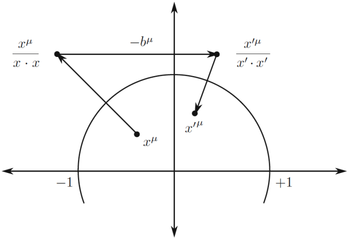

Figure 1: Illustration of a finite SCT

Conformal algebra

The generators of conformal transformations are: $P^\mu = i\partial^\mu$ (translation), $M^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)$ (rotation), $D = i x_\mu \partial^\mu$ (dilation or scaling), and $\mathfrak{K}^\mu = i\left(2x^\mu x_\nu \partial^\nu - x^2 \partial^\mu\right)$ (SCT).
Therefore, the full Conformal algebra is given by

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0;\ [\mathfrak{K}_\mu, \mathfrak{K}_\nu] = 0; \\
[D, P_\mu] &= -iP_\mu;\ [D, \mathfrak{K}_\mu] = i\mathfrak{K}_\mu; \\
[P_\rho, M_{\mu\nu}] &= i\left(g_{\rho\mu} P_\nu - g_{\rho\nu} P_\mu\right); \\
[\mathfrak{K}_\rho, M_{\mu\nu}] &= i\left(g_{\rho\mu} \mathfrak{K}_\nu - g_{\rho\nu} \mathfrak{K}_\mu\right); \\
[M_{\alpha\beta}, M_{\rho\sigma}] &= -i\left(g_{\beta\sigma} M_{\alpha\rho} - g_{\beta\rho} M_{\alpha\sigma} + g_{\alpha\rho} M_{\beta\sigma} - g_{\alpha\sigma} M_{\beta\rho}\right); \\
[\mathfrak{K}_\mu, P_\nu] &= -2i\left(g_{\mu\nu} D + M_{\mu\nu}\right);\ [D, M_{\mu\nu}] = 0.
\end{aligned}
$$

Boost as Rotations in 4D

The 4D Angular momentum tensor is given by:

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) = \begin{pmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & J^3 & -J^2 \\ K^2 & -J^3 & 0 & J^1 \\ K^3 & J^2 & -J^1 & 0 \end{pmatrix}. \tag{3.1}$$

$$\left[M^{\alpha\beta}, M^{\rho\sigma}\right] = -i\left(g^{\beta\sigma} M^{\alpha\rho} - g^{\beta\rho} M^{\alpha\sigma} + g^{\alpha\rho} M^{\beta\sigma} - g^{\alpha\sigma} M^{\beta\rho}\right) \tag{3.2}$$

There are $\frac{n(n-1)}{2}$ number of planes in n dimension:

Dimension	# of planes
1D	0
2D	1
3D	3
4D	6
5D	10
6D	15

(3.3)

Manifestly Covariant Conformal Algebra $(3+1)$

We define the following 6×6 tensor in the projective-space-time:

$$J_{a,b} = \begin{pmatrix} 0 & -D & -\frac{\mathfrak{K}_0}{\sqrt{2}} & -\frac{\mathfrak{K}_1}{\sqrt{2}} & -\frac{\mathfrak{K}_2}{\sqrt{2}} & -\frac{\mathfrak{K}_3}{\sqrt{2}} \\ D & 0 & \frac{P_0}{\sqrt{2}} & \frac{P_1}{\sqrt{2}} & \frac{P_2}{\sqrt{2}} & \frac{P_{\hat{3}}}{\sqrt{2}} \\ \frac{\mathfrak{K}_0}{\sqrt{2}} & -\frac{P_0}{\sqrt{2}} & 0 & \mathcal{D}^{\hat{1}} & \mathcal{D}^{\hat{2}} & K^3 \\ \frac{\mathfrak{K}_1}{\sqrt{2}} & -\frac{P_1}{\sqrt{2}} & -\mathcal{D}^{\hat{1}} & 0 & J^3 & -\mathcal{K}^{\hat{1}} \\ \frac{\mathfrak{K}_2}{\sqrt{2}} & -\frac{P_2}{\sqrt{2}} & -\mathcal{D}^{\hat{2}} & -J^3 & 0 & -\mathcal{K}^{\hat{2}} \\ \frac{\mathfrak{K}_{\hat{3}}}{\sqrt{2}} & -\frac{P_{\hat{3}}}{\sqrt{2}} & -K^3 & \mathcal{K}^{\hat{1}} & \mathcal{K}^{\hat{2}} & 0 \end{pmatrix}_{(6\times 6)} \tag{3.4}$$

Then, the simplified conformal algebra $SO(4+1,1)$ is:

$$[J_{ab}, J_{cd}] = -i\left(g_{bd}J_{ac} - g_{bc}J_{ad} + g_{ac}J_{bd} - g_{ad}J_{bc}\right) \tag{3.5}$$

where,

$$g_{\hat{a},\hat{b}} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}_{6\times 6} \tag{3.6}$$

where $a, b \in \{-2, -1, 0, 3\}$.

Isomorphism with Dirac matrices

Dirac[1] has shown the existence of isomorphism between $SO(4,2)$ conformal group and Dirac matrices. Later, Hepner[2] has explicitly shown the isomorphism between the group of Dirac's four-row γ-matrices and the continuous conformal group in Euclidean space.
We show

$$J'_{a,b} = \begin{pmatrix} 0 & \gamma_5 & \frac{(1-\gamma_5)\gamma_0}{\sqrt{2}} & \frac{(1-\gamma_5)\gamma_1}{\sqrt{2}} & \frac{(1-\gamma_5)\gamma_2}{\sqrt{2}} & \frac{(1-\gamma_5)\gamma_3}{\sqrt{2}} \\ -\gamma_5 & 0 & \frac{(1+\gamma_5)\gamma_0}{\sqrt{2}} & \frac{(1+\gamma_5)\gamma_1}{\sqrt{2}} & \frac{(1+\gamma_5)\gamma_2}{\sqrt{2}} & \frac{(1+\gamma_5)\gamma_3}{\sqrt{2}} \\ \frac{-(1-\gamma_5)\gamma_0}{\sqrt{2}} & \frac{-(1+\gamma_5)\gamma_0}{\sqrt{2}} & 0 & \gamma_0\gamma_1 & \gamma_0\gamma_2 & \gamma_0\gamma_3 \\ \frac{-(1-\gamma_5)\gamma_1}{\sqrt{2}} & \frac{-(1+\gamma_5)\gamma_1}{\sqrt{2}} & \gamma_1\gamma_0 & 0 & \gamma_1\gamma_2 & \gamma_1\gamma_3 \\ \frac{-(1-\gamma_5)\gamma_2}{\sqrt{2}} & \frac{-(1+\gamma_5)\gamma_2}{\sqrt{2}} & \gamma_2\gamma_0 & \gamma_2\gamma_1 & 0 & \gamma_2\gamma_3 \\ \frac{-(1-\gamma_5)\gamma_3}{\sqrt{2}} & \frac{-(1+\gamma_5)\gamma_3}{\sqrt{2}} & \gamma_3\gamma_0 & \gamma_3\gamma_1 & \gamma_3\gamma_2 & 0 \end{pmatrix}. \tag{3.7}$$

This $J'_{a,b}$ obeys the $SO(4+1,1)$ algebra
$[J_{ab}, J_{cd}] = -i\left(g_{bd}J_{ac} - g_{bc}J_{ad} + g_{ac}J_{bd} - g_{ad}J_{bc}\right)$

[1] Dirac, P. A. M. Annals Math. 37, 429–442 (1936)
[2] Hepner, W. A. Nuovo Cim. 26, 351–368 (1962).

Isomorphism with Dirac matrices

The representation of the conformal group in terms of 4×4 gamma matrices are the following;

$$P_\mu = \frac{i}{2}(1+\gamma_5)\gamma_\mu; \tag{3.8}$$

$$\mathfrak{K}_\mu = \frac{-i}{2}(1-\gamma_5)\gamma_\mu; \tag{3.9}$$

$$K^1 = \frac{i}{2}\gamma_1\gamma_0; \quad K^2 = \frac{i}{2}\gamma_2\gamma_0; \quad K^3 = \frac{i}{2}\gamma_3\gamma_0; \tag{3.10}$$

$$J^1 = \frac{i}{2}\gamma_2\gamma_3; \quad J^2 = \frac{i}{2}\gamma_3\gamma_1; \quad J^3 = \frac{i}{2}\gamma_1\gamma_2; \tag{3.11}$$

$$D = \frac{-i}{2}\gamma_5. \tag{3.12}$$

Manifestly Covariant Conformal Algebra $(1+1)$

We have:

$$J_{ab} = \begin{pmatrix} 0 & -D & -\frac{\mathfrak{K}_0}{\sqrt{2}} & -\frac{\mathfrak{K}_3}{\sqrt{2}} \\ D & 0 & \frac{P_0}{\sqrt{2}} & \frac{P_3}{\sqrt{2}} \\ \frac{\mathfrak{K}_0}{\sqrt{2}} & -\frac{P_0}{\sqrt{2}} & 0 & -K^3 \\ \frac{\mathfrak{K}_3}{\sqrt{2}} & -\frac{P_3}{\sqrt{2}} & K^3 & 0 \end{pmatrix}_{(4\times 4)} \tag{4.1}$$

Then, the $1+1$ conformal algebra in LFD is given by

$$[J_{ab}, J_{cd}] = -i\left(g_{bd}J_{ac} - g_{bc}J_{ad} + g_{ac}J_{bd} - g_{ad}J_{bc}\right) \tag{4.2}$$

where,

$$g_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{4\times 4} \tag{4.3}$$

From (4.2), one can write

$$J_{ab} = i(x_a\partial_b - x_b\partial_a) \tag{4.4}$$

where $a, b \in \{-2, -1, 0, 3\}$.

Manifestly Covariant Conformal Algebra $(1 + 1)$

Let's say that A^a is hyper-4-vector; suppose A^a and B^a transform under 6d rotation:

$$A'^a = R^a{}_b A^b, \quad B'^a = R^a{}_b B^b. \tag{4.5}$$

Then the inner products $A' \cdot B'$ and $A \cdot B$ can be written as

$$A'_b B'^b = (g_{ab} R^a{}_c R^b{}_d) A^c B^d, \tag{4.6}$$

$$A_b B^b = g_{cd} A^c B^d. \tag{4.7}$$

In order for $A'.B' = A.B$ to hold for any A and B, the coefficients of $A^c B^d$ should be the same term by term:

$$g_{ab} R^a{}_c R^b{}_d = g_{cd}. \tag{4.8}$$

Let's start by looking at a hyper-4d rotation transformation, which is (has to be) infinitesimally close to the identity:

$$R^a{}_b = g^a{}_b + \omega^a{}_b \quad , \tag{4.9}$$

where $\omega^a{}_b$ is a set of small (real) numbers.

Inserting this into the defining condition, we get

$$\begin{aligned} g^{cd} &= R^c{}_b R^{bd} \ , && (4.10) \\ &= (g^c{}_b + \omega^c{}_b)(g^{bd} + \omega^{bd}) \ , \\ &= g^{cd} + \omega^{cd} + \omega^{dc} + \mathcal{O}(\omega^2). && (4.11) \end{aligned}$$

Keeping terms to the first order in ω, we then obtain

$$\omega^{ab} = -\omega^{ba} \ . \qquad (4.12)$$

Thus, it has 6 independent parameters:

$$\omega^{ab} = \begin{pmatrix} 0 & \omega^{-2,-1} & \omega^{-2,0} & \omega^{-2,3} \\ -\omega^{-2,-1} & 0 & \omega^{-1,0} & \omega^{-1,3} \\ -\omega^{-2,0} & -\omega^{-1,0} & 0 & \omega^{0,3} \\ -\omega^{-2,3} & -\omega^{1,3} & -\omega^{0,3} & 0 \end{pmatrix}_{(4\times 4)} \ . \qquad (4.13)$$

This can be conveniently parameterized using 6 anti-symmetric matrices as

$$\omega^{ab} = -i \sum_{c<\hat{d}} \omega^{cd} (J_{cd})^{ab} \ , \qquad (4.14)$$

where

$$(J_{cd})^{ab} = i(g^a{}_c g^b{}_d - g^b{}_c g^a{}_d). \tag{4.15}$$

then

$$\omega^{ab} = -i \sum_{c<d} \omega^{cd} (J_{cd})^{ab} = -i \sum_{c>d} \omega^{cd} (J_{cd})^{ab} = -\frac{i}{2} \omega^{cd} (J_{cd})^{ab} \ , \tag{4.16}$$

where in the last expression, the sum over all values of $\hat{c}$ and $\hat{d}$ is implied. The infinitesimal transformation (hyper-4d rotation) ((4.9)) can then be written a

$$R^a{}_b = g^a{}_b - \frac{i}{2} \omega^{cd} (J_{cd})^a{}_b \ , \tag{4.17}$$

The generator representation $(J_{cd})^a{}_b$ can be obtained by

$$(J_{cd})^a{}_b = (J_{cd})^{af} g_{fb} \tag{4.18}$$

Manifestly Covariant Conformal Algebra $(1+1)$

The representation matrices of conformal generators are defined by taking the first index to be superscript and the second subscript:

$$\frac{-\mathfrak{K}_\mu}{\sqrt{2}} \equiv (J_{-2\mu})^a{}_b \ ; \quad \frac{P_\mu}{\sqrt{2}} \equiv (J_{-1\mu})^a{}_b \ ; \quad -D \equiv (J_{-2-1})^a{}_b \ ; \quad K^3 \equiv (J_{3,0})^a{}_b . \tag{4.19}$$

where $a, b \in \{-2, -1, 0, 3\}$ and $\mu \in \{0, 3\}$.
Explicitly[3],

$$\mathfrak{K}_0 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \quad \mathfrak{K}_3 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} ; \quad D = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_0 = \sqrt{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \quad P_3 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} ; \quad K^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \tag{4.20}$$

[3] H. Ravikumar, Chueng Ji. to be published

Manifestly Covariant Conformal Algebra $(1+1)$

We define the conformal transformations as those transformations preserving the light cone. This is equivalent to preserving angles, and also equivalent to preserving ratios of lengths. Let's consider a hyper-4d vector;

$$\tilde{x}_{-1} = \frac{-\lambda}{\sqrt{2}}; \tag{4.21}$$

$$\tilde{x}_{-2} = \frac{-\lambda}{\sqrt{2}}(x^{\mu}.x_{\mu}); \tag{4.22}$$

$$\tilde{x}_{\mu} = \lambda x_{\mu}; \tag{4.23}$$

Now, let's consider the hyper-4d dot product,

$$g_{ab}\tilde{x}^{a}\tilde{x}^{b} = \tilde{x}^{-2}\tilde{x}_{-2} + \tilde{x}^{-1}\tilde{x}_{-1} + \tilde{x}^{\mu}\tilde{x}_{\mu} \tag{4.24}$$

$$\tilde{x}_{a}.\tilde{x}^{a} = -2\tilde{x}_{-2}\tilde{x}_{-1} + \tilde{x}^{\mu}\tilde{x}_{\mu} \qquad (\text{since, } \tilde{x}_{-1} = -\tilde{x}^{-2} \ \& \ \tilde{x}_{-2} = -\tilde{x}^{-1}) \tag{4.25}$$

$$\tilde{x}_{a}.\tilde{x}^{a} = -2\frac{-\lambda}{\sqrt{2}}(x^{\mu}.x_{\mu})\frac{-\lambda}{\sqrt{2}} + \lambda^{2}x^{\mu}x_{\mu} = -\lambda^{2}x^{\mu}x_{\mu} + \lambda^{2}x^{\mu}x_{\mu} \tag{4.26}$$

$$\tilde{x}_{a}.\tilde{x}^{a} = 0 \tag{4.27}$$

then,

$$x_\mu = -\frac{1}{\sqrt{2}}\frac{\tilde{x}_\mu}{\tilde{x}_{-1}} \tag{4.28}$$

$$\boxed{\frac{x_\mu}{x^2}} = -\frac{1}{\sqrt{2}}\frac{\tilde{x}_\mu}{\tilde{x}_{-1}}\frac{2\tilde{x}_{-1}\tilde{x}_{-1}}{\tilde{x}_\mu\tilde{x}^\mu} = -\frac{1}{\sqrt{2}}\frac{\tilde{x}_\mu}{\tilde{x}_{-1}}\frac{2\tilde{x}_{-1}\tilde{x}_{-1}}{2\tilde{x}_{-2}\tilde{x}_{-1}} = \boxed{-\frac{1}{\sqrt{2}}\frac{\tilde{x}_\mu}{\tilde{x}_{-2}}} \tag{4.29}$$

Also, $\boxed{x^2 = \frac{\tilde{x}_{-2}}{\tilde{x}_{-1}}}$.

Let's find the space-time transformation under each conformal generators in $1+1$.

For $\mathfrak{K}_0$:

$$\begin{pmatrix} \tilde{x}'_{-2} \\ \tilde{x}'_{-1} \\ \tilde{x}'_{0} \\ \tilde{x}'_{3} \end{pmatrix} = \exp\left(-ib^0 \mathfrak{K}_0\right) \begin{pmatrix} \tilde{x}_{-2} \\ \tilde{x}_{-1} \\ \tilde{x}_{0} \\ \tilde{x}_{3} \end{pmatrix} \tag{4.30}$$

$$\begin{pmatrix} \tilde{x}'_{-2} \\ \tilde{x}'_{-1} \\ \tilde{x}'_{0} \\ \tilde{x}'_{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ (b^0)^2 & 1 & \sqrt{2}b^0 & 0 \\ \sqrt{2}b^0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_{-2} \\ \tilde{x}_{-1} \\ \tilde{x}_{0} \\ \tilde{x}_{3} \end{pmatrix} \tag{4.31}$$

$$\begin{pmatrix} \tilde{x}'_{-2} \\ \tilde{x}'_{-1} \\ \tilde{x}'_{0} \\ \tilde{x}'_{3} \end{pmatrix} = \begin{pmatrix} \tilde{x}_{-2} \\ (b^0)^2\tilde{x}_{-2} + \tilde{x}_{-1} + \sqrt{2}b^0\tilde{x}_0 \\ \sqrt{2}b^0\tilde{x}_{-2} + \tilde{x}_0 \\ \tilde{x}_3 \end{pmatrix} \tag{4.32}$$

Manifestly Covariant Conformal Algebra $(1+1)$

On space-time, this transformation gives

$$x'_0 = \frac{-1}{\sqrt{2}}\frac{\tilde{x}'_0}{\tilde{x}'_{-1}} \tag{4.33}$$

$$= \frac{-1}{\sqrt{2}}\frac{\sqrt{2}b^0\tilde{x}_{-2}+\tilde{x}_0}{(b^0)^2\tilde{x}_{-2}+\tilde{x}_{-1}+\sqrt{2}b^0\tilde{x}_0} \tag{4.34}$$

$$= \frac{-1}{\sqrt{2}}\frac{\sqrt{2}b^0\frac{\tilde{x}_{-2}}{\tilde{x}_{-1}}+\frac{\tilde{x}_0}{\tilde{x}_{-1}}}{(b^0)^2\frac{\tilde{x}_{-2}}{\tilde{x}_{-1}}+1+\sqrt{2}b^0\frac{\tilde{x}_0}{\tilde{x}_{-1}}} \tag{4.35}$$

$$= -\frac{1}{\sqrt{2}}\frac{\sqrt{2}b^0x^2+\left(-\sqrt{2}x_0\right)}{(b^0)^2x^2+1+\sqrt{2}b^0\left(-\sqrt{2}x_0\right)} \tag{4.36}$$

$$= \frac{-b^0x^2+x_0}{(b^0)^2x^2+1-2b^0x_0} \tag{4.37}$$

$$x'_0 = \frac{x_0-b_0x^2}{1-2b^0x_0+(b^0)^2x^2}\ \checkmark \tag{4.38}$$

Transformation of x_0 and x_3 under conformal transformation

In the instant form:

Generators	x'_0	x'_3
$\mathfrak{K}_0$	$\boxed{x'_0 = \frac{x_0 - b_0 x^2}{1 - 2b_0 x_0 + (b_0)^2 x^2}}$	$\boxed{x'_3 = \frac{x_3}{1 - 2b_0 x_0 + (b_0)^2 x^2}}$
$-\mathfrak{K}_3$	$\boxed{x'_0 = \frac{x_0}{1 - 2b_3 x_{\underline{\hat{}}} - (b_3)^2 x^2}}$	$\boxed{x'_3 = \frac{x_3 + b_3 x^2}{1 - 2b_3 x_3 - (b_3)^2 x^2}}$
P_0	$\boxed{x'_0 = x_0 + a_0}$	$\boxed{x'_3 = x_3}$
$-P_3$	$\boxed{x'_0 = x_0}$	$\boxed{x'_3 = x_3 - a_3}$
D	$\boxed{x'_0 = e^{-\alpha} x_0}$	$\boxed{x'_3 = e^{-\alpha} x_3}$
K^3	$\boxed{x'_0 = (\cosh \eta_3) x_0 - (\sinh \eta_3) x_3}$	$\boxed{x'_3 = -(\sinh \eta_3) x_0 + (\cosh \eta_3) x_3}$

Combining D and K^3

Under boosts along the z-axis,

$$\begin{bmatrix} x'^0 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{bmatrix} \begin{bmatrix} x^0 \\ x^3 \end{bmatrix} \tag{4.39}$$

this gives,

$$x'^0 = x^0 \cosh\eta - x^3 \sinh\eta \tag{4.40}$$

$$x'^3 = -x^0 \sinh\eta + x^3 \cosh\eta \tag{4.41}$$

then the light-front coordinates under the boost,

$$x'^{\pm} = \frac{(x'^0 \pm x'^3)}{\sqrt{2}} = \frac{((x^0 \cosh\eta - x^3 \sinh\eta)) \pm (-x^0 \sinh\eta + x^3 \cosh\eta)))}{\sqrt{2}} \tag{4.42}$$

$$= (\cosh\eta \mp \sinh\eta) x^{\pm} \tag{4.43}$$

$$x'^{\pm} = e^{\mp\eta} x^{\pm} \tag{4.44}$$

Thus boost along the x^3-axis becomes a **scale transformation** for the variables x'^+ & x'^- and $x^+ = 0$ plane is invariant under the boost along x^3-axis.

Combining D and K^3

In IFD $(1+1)$, we have:

Table 1: $1+1$ conformal algebra in IFD

	P_0	$\mathfrak{K}_0$	D	$-P_3$	$\mathfrak{K}_3$	K^3
P_0	0	$2iD$	iP_0	0	$2iK^3$	$-iP_3$
$\mathfrak{K}_0$	$-2iD$	0	$-i\mathfrak{K}_0$	$-2iK^3$	0	$-i\mathfrak{K}_3$
D	$-iP_0$	$i\mathfrak{K}_0$	0	iP_3	$i\mathfrak{K}_3$	0
$-P_3$	0	$2iK^3$	$-iP_3$	0	$2iD$	iP_0
$\mathfrak{K}_3$	$-2iK^3$	0	$-i\mathfrak{K}_3$	$-2iD$	0	$-i\mathfrak{K}_0$
K^3	iP_3	$i\mathfrak{K}_3$	0	$-iP_0$	$i\mathfrak{K}_0$	0

Combining D and K^3

In LFD,

Table 2: $1+1$ conformal algebra in LFD[4]

	P_+	$\mathfrak{K}_+$	D_+	P_-	$\mathfrak{K}_-$	D_-
P_+	0	$2\sqrt{2}iD_+$	$\sqrt{2}iP_+$	0	0	0
$\mathfrak{K}_+$	$-2\sqrt{2}iD_+$	0	$-\sqrt{2}i\mathfrak{K}_+$	0	0	0
D_+	$-\sqrt{2}iP_+$	$\sqrt{2}i\mathfrak{K}_+$	0	0	0	0
P_-	0	0	0	0	$2\sqrt{2}iD_-$	$\sqrt{2}iP_-$
$\mathfrak{K}_-$	0	0	0	$-2\sqrt{2}iD_-$	0	$-\sqrt{2}i\mathfrak{K}_-$
D_-	0	0	0	$-\sqrt{2}iP_-$	$\sqrt{2}i\mathfrak{K}_-$	0

where, $P_\pm = \frac{P_0 \pm P_3}{\sqrt{2}}$, $\mathfrak{K}_\pm = \frac{\mathfrak{K}_0 \mp \mathfrak{K}_3}{\sqrt{2}}$, and $D_\pm = \frac{D \mp K^3}{\sqrt{2}}$.

[4] H. Ravikumar, Chueng Ji. to be published

Combining D and K^3

We find[5] that $SO(2,2)$ splits into a direct sum of two identical algebras:

$$SO(2,2) \simeq SO(1,2) \oplus SO(1,2) \tag{4.45}$$

Lets make two new 3×3 anti symmetric tensors, namely J^+_{ab} and J^-_{ab}. Where, $a, b \in \{0, 1, 2\}$.

$$J^+_{ab} = \frac{1}{2}\begin{pmatrix} 0 & P_+ & D - K^3 \\ -P_+ & 0 & \mathfrak{K}_- \\ -D + K^3 & -\mathfrak{K}_- & 0 \end{pmatrix}_{3\times 3} \quad ; \qquad J^-_{ab} = \frac{1}{2}\begin{pmatrix} 0 & P_- & D + K^3 \\ -P_- & 0 & \mathfrak{K}_+ \\ -D - K^3 & -\mathfrak{K}_+ & 0 \end{pmatrix}_{3\times 3} \tag{4.46}$$

and with the new metric g_{ab},

$$g_{ab} = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}_{3\times 3} \tag{4.47}$$

they fulfill the $SO(1,2)$ commutation relations;

$$\left[J^\pm_{ab}, J^\pm_{cd}\right] = -i\left(g_{bd}J^\pm_{ac} - g_{bc}J^\pm_{ad} + g_{ac}J^\pm_{bd} - g_{ad}J^\pm_{bc}\right); \qquad \left[J^+_{ab}, J^-_{cd}\right] \quad = 0$$

These are perfectly consistent with the above table in the LFD limit.

[5] Daniel Meise's Relations between 2D and 4D Conformal Quantum Field Theory